Exact Self-Dual Skyrmions from Conformal and Rational Map Ansätze

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Self-dual solutions and Extended Skyrme Model

Introduction

The Skyrme model [1,3] is an effective nonlinear theory of pions in 3+1 dimensions in the regime of low energy, with the Skyrme field $U(\mathbf{x},t) \in SU(2)$. An extension of this model was proposed in [4] where a real symmetric 3×3 matrix h coupled with the quadratic term and h^{-1} couplet to the quartic term of the Lagrangian generalize the standard model and give an auto-dual sector [5]. In models that have a dual-sector one can look for stable static solutions through self-duality equations, which are first order equations in the fields that automatically imply in the Euler-Lagrange equations and saturate the lower bound of the Bogomolny inequality. In this work we aim to understand the behavior of h in the self-dual sector and find field configurations for any integer topological charge.

Self-Duality

Let be $A_\alpha = A_\alpha(\chi_a, \partial_j \chi_b)$ and $\tilde{A}_\alpha = \tilde{A}_\alpha(\chi_a, \partial_j \chi_b)$ for the fields $\chi_a,\ a = 1,\dots N,\ j = 1,\dots,d$ the topological charge defined above is homotopicaly invariant ($\delta Q = 0$)

Topological charge $$Q := \int d^d x A_\alpha \tilde{A}_\alpha$$

Static Energy $$E = \int d^d x \left[A_\alpha^2 + \tilde{A}_\alpha^2\right] = \int d^d x \left[A_\alpha \mp \tilde{A}_\alpha\right]^2 \pm \int d^d x A_\alpha \tilde{A}_\alpha \geq \pm Q$$

Homot. inv. $$\delta Q = 0 \Rightarrow \partial_j \left(A_\alpha \frac{\delta \tilde{A}_\alpha}{\delta \partial_j \chi_a}\right) - A_\alpha \frac{\delta \tilde{A}_\alpha}{\delta \chi_a} + \partial_j \left(\tilde{A}_\alpha \frac{\delta A_\alpha}{\delta \partial_j \chi_a}\right) - \tilde{A}_\alpha \frac{\delta A_\alpha}{\delta \chi_a} = 0$$

Physical eq. $$\delta S = 0 \Rightarrow \partial_j \left(A_\alpha \frac{\delta A_\alpha}{\delta \partial_j \chi_a}\right) - A_\alpha \frac{\delta A_\alpha}{\delta \chi_a} + \partial_j \left(\tilde{A}_\alpha \frac{\delta \tilde{A}_\alpha}{\delta \partial_j \chi_a}\right) - \tilde{A}_\alpha \frac{\delta \tilde{A}_\alpha}{\delta \chi_a} = 0$$

Self-duality equation $A_\alpha = \pm \tilde{A}_\alpha \Longrightarrow E = |Q|$ **Lower bound**

Extended Skyrme Model[1]

$$S = \int d^4 x \left[\frac{m_0^2}{2} h_{ab} R_\mu^a R^{b,\mu} - \frac{1}{4e_0^2} h_{ab}^{-1} H_{\mu\nu}^a H^{b,\mu\nu}\right]$$

where m_0 e e_0 are coupling constants, $H_{\mu\nu}^a := \partial_\mu R_\nu^a - \partial_\nu R_\mu^a$, h is an inversible, symmetric and real 3×3 matrix. R_μ^a are the components of the $SU(2)$ Maurer-Cartan form give by

$$R_\mu = i\partial_\mu U U^\dagger \equiv R_\mu^a T_a \implies \partial_\mu R_\nu - \partial_\nu R_\mu + i[R_\mu, R_\nu] = 0$$ **Maurer-Cartan eq.**

$\forall U \in SU(2)$ and $T_a,\ a = 1,2,3,$ being the generators of the corresponding Lie algebra

$$[T_a, T_b] = i\varepsilon_{abc} T_c, \qquad \mathrm{Tr}\,(T_a T_b) = \kappa \delta_{ab}$$

Symmetry $SU(2)_L \otimes SU(2)_R$

- **Right** $U \to U g_R;\quad R_\mu^a \to R_\mu^a;\quad h_{ab} \to h_{ab}$
- **Left** $U \to g_L U;\quad R_\mu^a \to d_{ab}(g_L) R_\mu^b;\quad h_{ab} \to d_{ab}(g_L) h_{bc} d_{cd}^T(g_L)$

$g T_a g^{-1} = T_b d_{ba}(g)$. The solutions are classified by the winding number of the map that is a integer since $\pi_3(S^3) = \mathbb{Z}$ with the following integral representation $S^3 \to SU(2) \equiv S^3$

$$Q = \frac{i}{48\kappa\pi^2} \int d^3 x \varepsilon_{ijk}\, \mathrm{Tr}\,(R_i R_j R_k) = -\frac{1}{48\kappa\pi^2} \int d^3 x R_i^a \varepsilon_{ijk} \partial_j R_k^a \qquad h = kk^T$$

$$A_i^a \equiv R_i^b k_{ba} \qquad \tilde{A}_i^a \equiv k_{ab}^{-1} \varepsilon_{ijk} \partial_j R_k^b k_{ba}$$

Self-duality equation

$$\lambda h_{ab} R_i^b = \frac{1}{2} \varepsilon_{ijk} H_{ij}^a \quad \text{or} \quad \vec{\nabla} \wedge \vec{R}_a = \lambda h_{ab} \vec{R}_b, \qquad \lambda \equiv \pm m_0 e_0$$

Remarkable new results:

$$R_{ia} \equiv R_i^a \Longrightarrow Q = -\frac{\lambda^3}{16\pi^2} \int d^3 x \det h, \qquad h = \frac{\det R}{\lambda} \left(R^T R\right)^{-1}$$

Rational Map Ansatz

Parameterize $U = W^\dagger e^{ifT_3/2} W, \quad W \equiv \frac{1}{\sqrt{1+|u|^2}} \begin{pmatrix} 1 & iu \\ i\bar{u} & 1 \end{pmatrix}, \quad V \equiv W^\dagger e^{ifT_3/2}$

Coordinates $x_1 = r\frac{-i(w-\bar{w})}{1+|w|^2}, \quad x_2 = r\frac{(w+\bar{w})}{1+|w|^2}, \quad x_3 = r\frac{|w|^2-1}{1+|w|^2}$

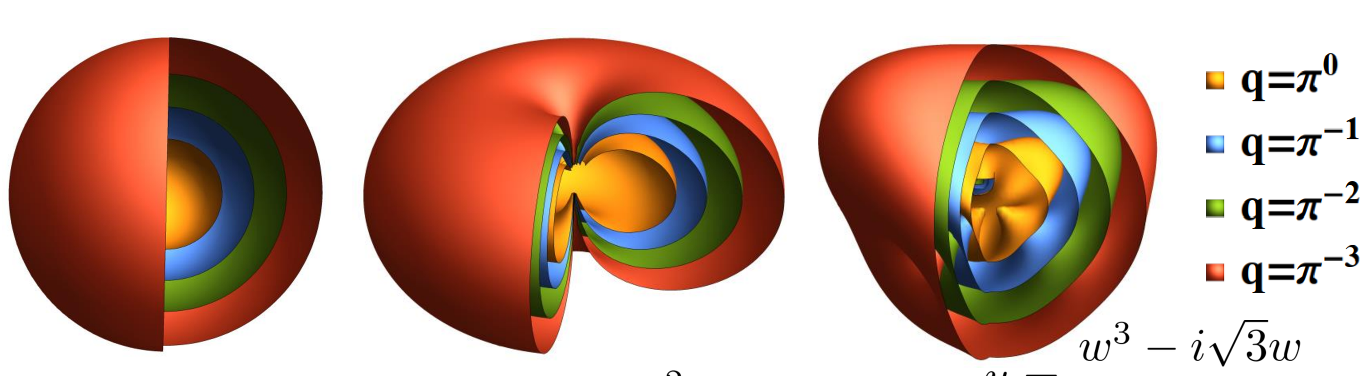
The holomorphic ansatz $f \equiv f(r),\ u = u(w),\ \bar{u} = \bar{u}(\bar{w})$ diagonalize $\tilde{h}_{ab} \equiv d_{ac}(V^\dagger) h_{cd} d_{db}^T(V^\dagger)$

Rational map ansatz $u(w) = p(w)/g(w)$, p e g are polynomial of degree n and m.

Charge $Q = \max(m,n)\left[f(r) - \sin f(r)\right]_0^\infty$
Boundary cond. $\forall f \in [0, 2\pi N]$

ISOSURFACES OF CHARGE DENSITY

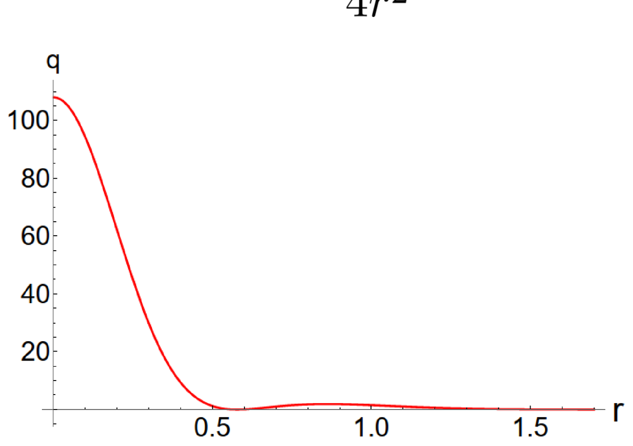
$$f(r) = 4\arctan\left(\frac{a}{r}\right)$$



$u = w \qquad u = w^2 \qquad u = \frac{w^3 - i\sqrt{3}w}{i\sqrt{3}w^2 - 1}$

CHARGE DENSITY (u=w, N=3)

$$q_Q = -\frac{N f' \sin^2(Nf/2)}{4r^2}$$



Toroidal Ansatz

Parameterize $U = \begin{pmatrix} Z_2 & i Z_1 \\ i\bar{Z}_1 & \bar{Z}_2 \end{pmatrix}, \qquad |Z_1|^2 + |Z_2|^2 = 1$

Coordinates $x_1 = \frac{a}{p}\sqrt{z}\cos\varphi, \quad x_2 = \frac{a}{p}\sqrt{z}\sin\varphi, \quad x_1 = \frac{a}{p}\sqrt{1-z}\sin\xi, \quad z = \frac{4a^2(x_1^2+x_2^2)}{(x_1^2+x_2^2+x_3^2+a^2)^2}$

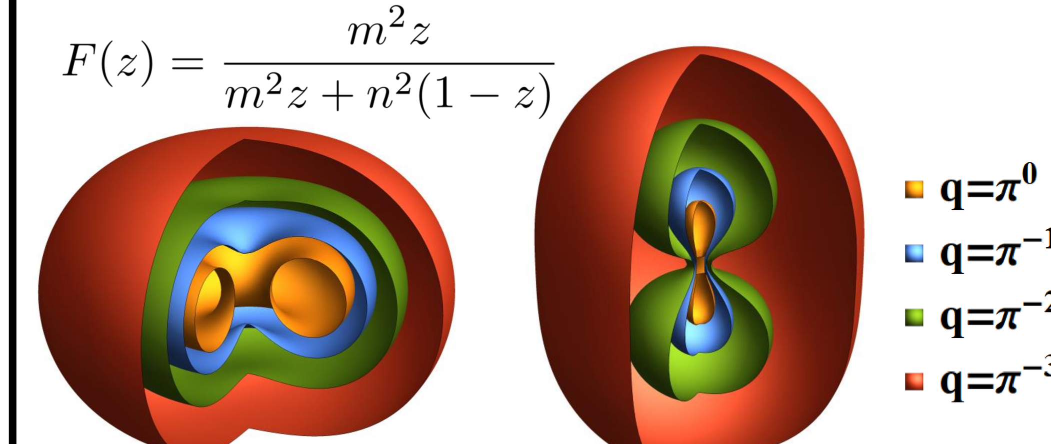
$0 \leq \varphi, \xi \leq 2\pi, \quad 0 \leq z \leq 1, \quad p = 1 - \sqrt{1-z}\cos\xi,$

Toroidal ansatz $Z_1 = \sqrt{F(z)}\, e^{i\,n\,\varphi}, \quad Z_2 = \sqrt{1-F(z)}\, e^{i\,m\,\xi}, \quad \forall m, n \in \mathbb{Z}$

Charge $Q = mn\left[F(1) - F(0)\right]$ **Boundary cond.** $\forall F \in [0,1]$

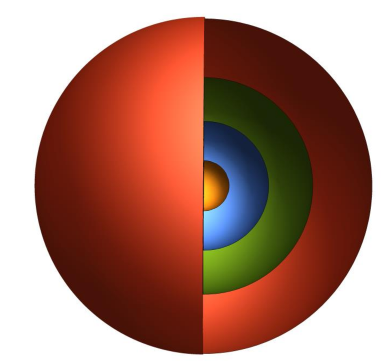
ISOSURFACES OF CHARGE DENSITY

$$F(z) = \frac{m^2 z}{m^2 z + n^2(1-z)}$$



$m = 1,\ n = 3 \qquad m = 5,\ n = 1$

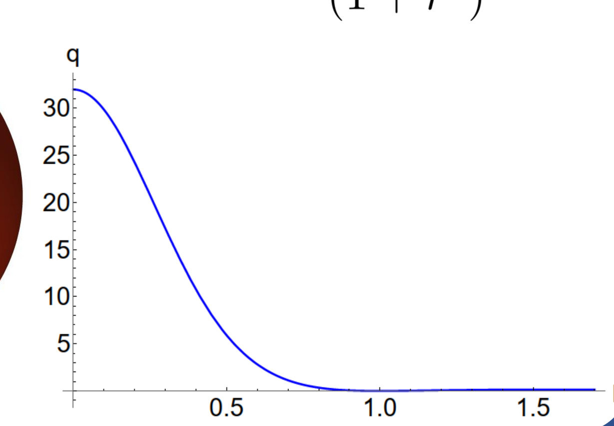
$$F(z) = z$$



$m = n = 1$

CHARGE DENSITY (F=z)

$$q_Q^{\text{reesc.}} = \frac{4mn}{(1+r^2)^3}$$



Generalized model

$$\begin{aligned} \widehat{S} \equiv & \int d^4 x \left[\frac{m_0^2}{2} h_{ab} R_\mu^a R^{b,\mu} - \frac{1}{4e_0^2} h_{ab}^{-1} H_{\mu\nu}^a H^{b,\mu\nu}\right. \\ & + \frac{\mu_0^2}{2} \mathrm{Tr}\,(\partial_\mu h)^2 + V(h_{ab}) - \gamma_1^2 \left[3 m_0^2 e_0^2 \det h + h_{ab} R_\mu^a R^{b,\mu}\right]^2 \\ & \left. - \gamma_2^2 \left[m_0^2 e_0^2 \det h\ \mathrm{Tr}\,(h^{-2}) + h_{ab}^{-1} R_\mu^a R^{b,\mu}\right]^2\right] \end{aligned}$$

Self-dual Ansatz

$U = W^\dagger e^{ifT_3/2} W, \quad G \equiv W^\dagger e^{ifT_3/2}$

$\tilde{h}_{ab}^{(diag.)} \equiv d_{ac}(V^\dagger) h_{cd} d_{db}^T(V^\dagger), \quad G = V^\dagger$

$\tilde{h}_{11} = \tilde{h}_{22} \equiv \varphi_1(r)$

$\tilde{h}_{33} \equiv \varphi_3(r) q$

$q \equiv \frac{(1+|w|^2)^2}{(1+|u|^2)^2} u' \bar{u}'$

$$+ \ V = \sum_{a=1}^{3} \left(\frac{\tilde{h}_{aa}^3}{6a} - \frac{\tilde{h}_{aa}^4}{8}\right)$$

Exact Q=1 solution!

$$\begin{cases} f(r) = 4\arctan(a/r) \\ \varphi_1 = \varphi_3 = 4a/(a^2+r^2) \\ u = w \Rightarrow q = 1 \end{cases}$$

Conclusion

In this work it has been demonstrated that h can be determined for each $U \in SU(2)$ that can be chosen arbitrarily, as long as it leaves h invertible and the charge integer. Through each of the ansätze, rational and toroidal map, we show how to construct solutions for any charge integer.

References

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